UUITP-06/97
10A 97-05
8 April 1997

Unitary matrix model for toroidal compactifications of M theory

Alexios P. Polychronakos[†]

Theoretical Physics Dept., University of Ioannina
45110 Ioannina, Greece

and

Theoretical Physics Dept., Uppsala University
S-751 08 Uppsala, Sweden

ABSTRACT

A unitary matrix model is proposed as the large-N matrix formulation of M theory on flat space with toroidal topology. The model reproduces the motion of elementary D-particles on the compact space, and admits membrane states with nonzero wrapping around nontrivial 2-tori even at finite N.

† poly@calypso.teorfys.uu.se

The recently proposed matrix model approach to M-theory consists of the dimensional reduction of 10-dimensional super-Yang-Mills (SYM) to 0+1 dimensions [1]. This model, which has appeared in the past in different contexts [2-4], has its origins in D-brane dynamics [5-9] and is known to describe the dynamics of D-particles in the low-energy (nonrelativistic) limit [10-12]. The remarkable conjecture made in [1] is that the full light-cone dynamics of the different physical objects within M-theory are imbedded in the large-N limit of the above matrix model. This conjecture has survived a number of consistency checks [13-19] in a rapidly increasing literature.

The above matrix model applies to the case of flat uncompactified spacetime and would need be modified for other topologies. For the case of toroidal compactifications of space, the model should account for the interactions due to virtual strings winding around the compact dimensions, as well as describe membranes wrapped around compact submanifolds. The main proposal for doing that is to enlarge the matrix model to a (K+1)-dimensional SYM filed theory, where K is the number of compact dimensions [1,20]. Although this certainly contains all the relevant degrees of freedom [14-21], it most probably contains more than is actually needed at the large-N limit. Furthermore, it takes away from the simplicity of the original matrix model by adding an infinity of degrees of freedom. This is somewhat undesirable. The hope would be that a matrix model with the correct dynamics will contain, in the large-N limit, all the relevant degrees of freedom. The purpose of this note is to point to a possible such model.

The proposed model consists of using *unitary* (rather than hermitian) matrices for each compact dimension. This proposal is an especially natural one in the context of D-brane dynamics. In fact, it is the eigenvalues of Wilson loop elements winding around compact dimensions of space and coupling to open strings which, in the dual picture, become the coordinates of D-branes [7]. In the uncompactified case we recover a description in terms of (the constant mode of) the gauge field. What we propose here is to preserve the Wilson element itself as the dynamical object and write an appropriate action which, in the large-N limit, recovers the

full physical spectrum of the theory.

For simplicity, we can assume that all 9 transverse dimensions are compactified with radii R_i. Then the Lagrangian would contain the terms

$$L = \mathrm{tr}\, R \left\{ \frac{R_I^2}{2} DU_i^\dagger DU_i + R_i^2 R_j^2 [U_i, U_j][U_i, U_j]^\dagger + \theta^T D\theta + R_i \theta^T \gamma^i U_i^\dagger [\theta, U_i] \right\} \quad (1)$$

as well as other possible higher-order terms. As usual, $DU_i = \dot{U}_i - i[A_0, U_i]$, and summation over all indices is assumed. A_0 implements the gauge invariance constraint

$$\sum_i R_i^2 (U_i^\dagger \dot{U}_i - \dot{U}_i U_i^\dagger) + \theta\theta^T = 0 \quad (2)$$

In the uncompactified limit $R_i \to \infty$ the usual hermitian matrix model is recovered upon writing $U_i = \exp(iX_i/R_i)$.

The above model in the low-energy limit describes the motion of D-particles on the compact space. The potential is minimized when the U_i commute; their eigenvalues are phases $\exp(i\phi_i)$, and $R_i\phi_i$ correspond to the coordinates of the D-particles on the spatial torus. It is easy to see that the kinetic term plus the gauge constraint for $\theta = 0$ imply free motion for the ϕ_i.

It is important that the present model can describe membranes wrapped around any two-torus in space even at finite N. In [1] such membrane solutions were identified, but the corresponding charges (winding numbers) were activated only at the infinite-N limit and were zero for any finite N. The construction is along similar lines as in [4,22-24]. The difference is that, now, the classical membrane coordinates are represented as $U_i(q,p) = \exp(iX_i(q,p)/R_i)$, where q, p are coordinates on the light-cone spatial world sheet of the membrane. U_i are periodic in q, p while X_i need not be. The membrane action is

$$A = \int dq dp (U_i^\dagger \partial_q U_i \; U_j^\dagger \partial_p U_j - U_i^\dagger \partial_p U_i \; U_j^\dagger \partial_q U_j)^2 \quad (3)$$

while the wrapping number of the membrane is

$$W = -\int dqdp(U_i^\dagger \partial_q U_i U_j^\dagger \partial_p U_j - U_i^\dagger \partial_p U_i U_j^\dagger \partial_q U_j) \tag{4}$$

Note that this would be zero if periodic X_i were used. Since U_i is periodic in p, q, its Fourier decomposition will be

$$U_i = \sum_{n,m} U_{i,nm}\ e^{iqn+ipm} \tag{5}$$

where the coefficients $U_{i,nm}$ will be constrained by the fact that U_i is a phase. For a smooth imbedding only the lowest several Fourier modes will be appreciable, so we can truncate the series to the lowest $N \times N$ coefficients (or, rather, periodically repete them beyond, which gives a discrete version of the membrane consisting of $N \times N$ points). From this, a corresponding matrix can be created

$$U_i = \sum_{n,m} U_{i,nm} U^q V^p \tag{6}$$

where U and V are the fundamental "quantum torus" coordinates, that is, $N \times N$ unitary matrices obeying $UV = \exp(i2\pi/N)VU$. In general, U_i will not be exactly unitary and a particular "normal ordering" will be required to unitarize it. (The same is true for the standard construction in terms of hermitian matrices.) This will modify the coefficients in (6) by terms of orden $1/N$. Expressions (3) and (4) for the membrane action and wrapping number are reproduced in the large-N limit as the real and imaginary part of the trace

$$A + iW = \mathrm{tr}(U_i U_j U_i^\dagger U_j^\dagger - 1) \tag{7}$$

This is exactly what appears as the potential term in the matrix model, which thus reproduces the membrane action. A stretched wrapping membrane will have

$U_i(q,p) = \exp(in_iq + im_ip)$ and thus corresponds to $U_i = U^{n_i}V^{m_i}$. For such matrices we get

$$U_iU_jU_i^\dagger U_j^\dagger = e^{i2\pi W/N} \tag{8}$$

where $W = \vec{n}_i \times \vec{n}_j = n_im_j - n_jm_i$ is the wrapping number. Therefore a better definition of the wrapping number would be the Z_N-phase of the SU(N) matrix $U_iU_jU_i^\dagger U_j^\dagger$. Clearly the wrapping is well-defined only when the matrix $U_iU_jU_i^\dagger U_j^\dagger$ has eigenvalues which are well-localized on the circle (their spread is, say, less than π). Also, it is defined modulo N. Both properties are relevant to the discretized finite-N description. Note, further, that we can construct many-membrane states, and thus reconstruct the full membrane Fock space in the large-N limit. For instance, the matrices

$$U_i = U_{N_1}^{n_1} \oplus U_{N_2}^{n_2}\ , \quad U_j = V_{N_1}^{m_1} \oplus V_{N_2}^{m_2} \tag{9}$$

(where U_N, V_N are the quantum torus matrices of dimensionality N), represent two stretched membranes with winding numbers n_1m_1 and n_2m_2 around the (i,j)-torus, and longitudinal momenta proportional to N_1 and N_2 respectively.

The matrix model potential has stationary points when

$$\sum_j U_iU_jU_i^\dagger U_j^\dagger = \sum_j U_j^\dagger U_iU_jU_i^\dagger \tag{10}$$

This has as only solutions the above many-membrane states we just considered. Thus, as expected, stretched membranes are local minima of the potential. The charge of a membrane wrapped along the longitudinal direction and cycle i, on the other hand, would be

$$W_i = \mathrm{tr}[\dot{U}_i(U_jU_i^\dagger U_j^\dagger - U_j^\dagger U_i^\dagger U_j)] \tag{11}$$

plus fermionic terms.

A crucial property of the matrix model description of M-theory is supersymmetry. The obvious generalization of the supersymmetry transformations for this model would be

$$
\begin{aligned}
\delta U_i &= (U_i\theta + \theta U_i)\gamma_i\epsilon \\
\delta\theta &= \frac{1}{2}(U_i^\dagger \dot{U}_i + \dot{U}_i U_i^\dagger + \gamma_{ij}(U_i U_j U_i^\dagger U_j^\dagger + c.p. - h.c.))\epsilon + \tilde{\epsilon}
\end{aligned}
\tag{12}
$$

For a static solution with $\theta = 0$ the above transformation will preserve half the SUSY when $U_i U_j U_i^\dagger U_j^\dagger$ is proportional to the identity, so that the dynamical part can be cancelled by the kinematical one [18]. This reproduces the stretched membrane solutions found earlier.

It can be checked, though, that the above is not an exact symmetry of the model. In principle, we could start with the bosonic part of the lagrangian and perform a "supersymmetric completion". This would in principle generate higher order terms in the potential (vanishing in the uncompactified limit) and corresponding extra terms in the SUSY transformations. We have not attempted the supersymmetrization of the model here, so this remains an open question. Potential problems reminiscent of fermion doubling and supersymmetry violation on the lattice might arise here, since our matrix model is essentially a dimensional reduction of 10-d SYM on the dual lattice. This would imply that the resulting SUSY model contains infinite terms. In that case it may not be different than the corresponding (K+1)-dimensional SYM theory, with all the field modes integrated out to give an effective action for the 'topological' degrees of freedom corresponding to Wilson loop elements around the nontrivial spatial cycles and fermionic zero modes. Even then, the implication is that these remaining degrees of freedom are enough in the large-N limit to describe the full spectrum of M theory. In fact, it has been suggested in [25] that supersymmetry may fix the potential ordering ambiguities in the matrix model that would describe D-brane motion on curved spaces (see also [26]). If the conjecture of the present note is correct, then, the M-theory matrix model for curved spaces would contain unitary matrices for spaces of

nontrivial connectivity, with the form of the action fixed by supersymmetry. This, and other consistency checks on the model, remain issues for investigation.

Acknowledgements: I would like to thank E. Floratos, V.P. Nair and K. Stelle for discussions. I would also like to thank Imperial College and Rockefeller University for their hospitality while this work was done.

REFERENCES

1. T. Banks, W. Fischler, S.H. Shenker and L. Susskind, hep-th/9610043.
2. M. Claudson and M.B. Halpern, *Nucl. Phys.* **B250** (1985) 689.
3. M. Baake, M. Reinicke and V. Rittenberg, *J. Math. Phys.* **26** (1985) 1070.
4. B. de Witt, J. Hoppe and H. Nicolai, *Nucl. Phys.* **B305** (1988) 545.
5. J. Dai, R.G. Leigh and J. Polchinski, *Mod. Phys. Lett. A* /bf A4 (1989) 2073.
6. R.G. Leigh, *Mod. Phys. Lett. A* **A4** (1989) 2767.
7. J. Polchinski, *Phys. Rev. Lett.* bf 75 (1995) 4724; TASI96 Lectures on D-branes, hepth/9611050.
8. C. Bachas, *Phys. Lett.* **B374** (1996) 37.
9. E. Witten, *Nucl. Phys.* **B460** (1996) 335.
10. U.H. Danielsson, G. Ferretti and B. Sundborg, *Int. Jour. Mod. Phys.* /bf A11 (1996) 5463.
11. D. Kabat and P. Pouliot, *Phys. Rev. Lett.* /bf 77 (1996) 1004.
12. M.R. Douglas, D. Kabat, P. Pouliot and S.H. Shenker, hep-th/9608024.
13. M. Berkooz and M.R. Douglas, hep-th/9610236.
14. L. Susskind, hep-th/9611164.
15. O.J. Ganor, S. Ramgoolam and W. Taylor, hep-th/9611202.
16. T. Banks, N. Seiberg and S.H. Shenker, hep-th/9612157.

17. N. Kim and S.-J. Rey, hepth/9701139.

18. T. Banks and N. Seiberg, hep-th/9702187.

19. M.R. Douglas, H. Ooguri and S.H. Shenker, hep-th/9702203.

20. W. Taylor, hep-th/9611042.

21. R. Dijkgraaf, E. Verlinde and H. Verlinde, hep-th/9703030.

22. J. Hoppe, *Int. Jour. Mod. Phys.* **A4** (1989) 5235.

23. D. Fairlie, P. Fletcher and C. Zachos, *J. Math. Phys.* **31** (1990) 1088.

24. E.G. Floratos, *Phys. Lett.* **B288** (1989) 335.

25. M.R. Douglas, hep-th/9703056.

26. A. Tseytlin, 9701125.